Homework 2
Algebra

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Proposition 0.1 (Exercise 1a)**.** *Let R be a commutative ring and $x \in R$ an element that is not a zero divisor. Let M be an R-module. Then*

$$\operatorname{Tor}_1^R(R/(x), M) \cong \{m \in M : xm = 0\}$$

Proof. Consider the sequence of R-modules

$$0 \longrightarrow R \xrightarrow{r \mapsto xr} R \xrightarrow{r \mapsto \overline{r}} R/(x) \longrightarrow 0$$

The map $r \mapsto xr$ is injective because x is not a zero divisor, so this is an exact sequence. Since R is free as an R-module, this is a projective resolution of $R/(x)$. Dropping the $R/(x)$ and applying the tensor functor $M \otimes -$ we get the complex

$$0 \longrightarrow M \otimes R \xrightarrow{m \otimes r \mapsto m \otimes (xr)} M \otimes R \longrightarrow 0$$

Note that $m \otimes (xr) = (xm) \otimes r$. We have a functorial isomorphism $M \otimes R \to M$ given by $m \otimes r \mapsto rm$, that is, the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes R & \xrightarrow{m \otimes r \mapsto (xm) \otimes r} & M \otimes R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m \otimes r \mapsto rm} & & \downarrow{\scriptstyle m \otimes r \mapsto rm} & & \\ 0 & \longrightarrow & M & \xrightarrow{m \mapsto xm} & M & \longrightarrow & 0 \end{array}$$

By definition, $\operatorname{Tor}_1(R/(x), M)$ is the homology of the top row tensor chain complex at the left copy of $M \otimes R$. Since the image is trivial, this homology is just the kernel of $m \otimes r \mapsto (xm) \otimes r$. Since the vertical maps are isomorphisms and the square commutes, this kernel is isomorphic to the kernel of $m \mapsto xm$. Thus

$$\operatorname{Tor}_1^R(R/(x), M) \cong \{m \in M : xm = 0\}$$

□

Proposition 0.2 (Exercise 1b)**.** *Let m, n be positive integers, and let $d = \gcd(n, m)$.*

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

Proof. We apply part (a) with $R = \mathbb{Z}$, $x = m$, and $M = \mathbb{Z}/n\mathbb{Z}$. Then

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \{x \in \mathbb{Z}/n\mathbb{Z} : mx = 0\}$$

This is the kernel of $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defined by $x \to mx$, so it is a subgroup of $\mathbb{Z}/n\mathbb{Z}$, so it is cyclic. It is generated by $\frac{n}{\gcd(n,m)}$, so it is cyclic of order $\gcd(n, m)$. □

Lemma 0.3 (for Exercises 2a,2b). *Let A be an abelian group. There is a free resolution of $\mathbb{Z}$-modules*

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \longrightarrow F \longrightarrow A \longrightarrow 0$$

In particular, this is a projective resolution of A.

Proof. Let $\{x_i\}_{i\in I}$ be a set of generators for A, and let F be the free abelian group generated by $\{x_i\}_{i\in I}$. Let $\pi : F \to A$ be the unique homomorphism such that $x_i \mapsto x_i$. Since $\ker \pi$ is a subgroup of a free abelian group, it is free. Let $\iota : \ker \pi \to F$ be the inclusion. Then the sequence

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \ker \pi \xrightarrow{\iota} F \xrightarrow{\pi} A \longrightarrow 0$$

is a free resolution of A. □

Proposition 0.4 (Exercise 2a). *Let A and B be abelian groups. Then* $\operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$ *for $n \geq 2$.*

Proof. Using the previous lemma, take a free resolution of A,

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \longrightarrow F \longrightarrow A \longrightarrow 0$$

Then $\operatorname{Tor}_n^{\mathbb{Z}}$ is the homology of the sequence

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \otimes B \longrightarrow F \otimes B \longrightarrow 0$$

All of the kernels are trivial except the two on the far right, so the 2nd homology and higher are zero. □

Proposition 0.5 (Exercise 2b). *Let A and B be abelian groups. Then* $\operatorname{Ext}_{\mathbb{Z}}^n(A, B) = 0$ *for $n \geq 2$.*

Proof. Using the previous lemma, take a free resolution of A,

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \longrightarrow F \longrightarrow A \longrightarrow 0$$

Then $\operatorname{Ext}_n^{\mathbb{Z}}$ is the homology of the sequence

$$0 \longrightarrow \operatorname{Hom}(F, B) \longrightarrow \operatorname{Hom}(E, B) \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

Starting with the 2nd homology, the kernel and image are both trivial, so $\operatorname{Ext}_n^{\mathbb{Z}} = 0$ for $n \geq 2$. □

Proposition 0.6 (Exercise 2c). *Let A be a finitely generated abelian group. Then A is free abelian if and only if* $\operatorname{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z}) = 0$.

Proof. First suppose that A is free abelian. Since A is finitely generated,

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}$$

Then

$$\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}}\left(\bigoplus_{i=1}^{n} \mathbb{Z}, \mathbb{Z}\right) \cong \prod_{i=1}^{n} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$

Since $\mathbb{Z}$ is a free module over itself, it is projective, so $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, B) = 0$ for any abelian group B, so $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$. Thus $\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \prod_{i=1}^{n} 0 = 0$.

Now suppose that $\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$. Since A is finitely generated,

$$A \cong \left(\bigoplus_{i=1}^{n} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{m} \mathbb{Z}/a_j\mathbb{Z}\right)$$

for some set of $a_j \in \mathbb{Z}$. Then

$$\begin{aligned} \operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) &\cong \operatorname{Ext}^1_{\mathbb{Z}}\left(\left(\bigoplus_{i=1}^{n} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{m} \mathbb{Z}/a_j\mathbb{Z}\right), \mathbb{Z}\right) \\ &\cong \left(\prod_{i=1}^{n} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})\right) \times \left(\prod_{j=1}^{m} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/a_j\mathbb{Z}, \mathbb{Z})\right) \end{aligned}$$

As already noted, $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$, so

$$\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \prod_{j=1}^{m} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/a_j\mathbb{Z}, \mathbb{Z})$$

However, we know that

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/a_j\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/a_j\mathbb{Z}$$

which is not zero unless $a_j = 1$. Thus since $\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$ by hypothesis, $a_j = 1$ for all j. That is,

$$A \cong \bigoplus_{i=1}^{m} \mathbb{Z}$$

so A is free abelian. □

Lemma 0.7 (for Exercise 3a)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be set maps so that $g \circ f$ is injective. Then f is injective.*

Proof. Let $a, b \in X$ so that $f(a) = f(b)$. Then $g \circ f(a) = g \circ f(b)$ which implies $a = b$ by injectivity of $g \circ f$. Hence f is injective. □

Proposition 0.8 (Exercise 3a)**.** *Fix a commutative ring R, and let*

$$0 \longrightarrow F' \xrightarrow{\ \phi\ } F \xrightarrow{\ \psi\ } F'' \longrightarrow 0$$

be an exact sequence of R-modules, where F'' is flat. Then F is flat if and only if F' is flat.

Proof. (All homomorphisms are R-module homomorphisms and all tensors are over R.) Let $\alpha : E' \to E$ be an injective homomorphism. Since F'' is flat, $F'' \otimes E' \to F'' \otimes E$ defined by $f'' \otimes e' \to f'' \otimes \alpha(e')$ is injective. Also, by Lemma 3.3 in Lang, the rows in the following diagram are exact. The diagram is commutative because the tensor product is a functor.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' \otimes E' & \longrightarrow & F \otimes E' & \longrightarrow & F'' \otimes E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' \otimes E & \longrightarrow & F \otimes E & \longrightarrow & F'' \otimes E & \longrightarrow & 0
\end{array}$$

Suppose F' is flat. Then $F' \otimes E' \to F' \otimes E$ is injective. Since $F'' \otimes E' \to F'' \otimes E$ is also injective and $0 \to 0$ is (trivially) surjective, the diagram satisfies the hypotheses of Exercise 15a from Chapter 3 which we did in Homework 1 (one of the Four Lemmas). Thus $F \otimes E' \to F \otimes E$ is injective. Thus F is flat.

Now suppose that F is flat. Then $F \otimes E' \to F \otimes E$ is injective, so the composition $F' \otimes E' \to F \otimes E' \to F \otimes E$ is a composition of injections, so it is injective. By commutativity of the left square, the composition $F' \otimes E' \to F' \otimes E \to F \otimes E$ must be injective. By the previous lemma, this implies that $F' \otimes E' \to F' \otimes E$ is injective, which means that F' is flat.

(Here's the same commutative diagram with detailed descriptions of the maps.)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' \otimes E' & \xrightarrow{f' \otimes e' \mapsto \phi(f') \otimes e'} & F \otimes E' & \xrightarrow{f \otimes e' \mapsto \psi(f) \otimes e'} & F'' \otimes E' & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle f' \otimes e' \mapsto f' \otimes \alpha(e')} & & \Big\downarrow {\scriptstyle f \otimes e' \mapsto f \otimes \alpha(e')} & & \Big\downarrow {\scriptstyle f'' \otimes e' \mapsto f'' \otimes \alpha(e')} & & \\
0 & \longrightarrow & F' \otimes E & \xrightarrow{f' \otimes e \mapsto \phi(f') \otimes e} & F \otimes E & \xrightarrow{f \otimes e \mapsto \psi(f) \otimes e} & F'' \otimes E & \longrightarrow & 0
\end{array}$$

□

Proposition 0.9 (Exercise 3a, alternate proof using a long exact sequence)**.** *Fix a commutative ring R, and let*

$$0 \longrightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \longrightarrow 0$$

be an exact sequence of R-modules, where F'' is flat. Then F is flat if and only if F' is flat.

Proof. Fix an R-module M and a projective resolution of M

$$\dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Then we have the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F' \otimes P_2 & \longrightarrow & F' \otimes P_1 & \longrightarrow & F' \otimes P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F \otimes P_2 & \longrightarrow & F \otimes P_1 & \longrightarrow & F \otimes P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F'' \otimes P_2 & \longrightarrow & F'' \otimes P_1 & \longrightarrow & F'' \otimes P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Each column is a short exact sequence, because P_i is projective and hence flat (Lang, Proposition 3.1(iii), page 613). That is, the above is a short exact sequence of complexes with morphisms of degree zero. For each row, the nth homology is $\mathrm{Tor}_n^R(F', M)$, $\mathrm{Tor}_n^R(F, M)$, or $\mathrm{Tor}_n^R(F'', M)$ respectively. By Theorem 2.1 (Lang, page 768), we get a long exact sequence on homology,

$$\begin{aligned}
\dots \to \mathrm{Tor}_n^R(F', M) \to \mathrm{Tor}_n^R(F, M) \to \mathrm{Tor}_n^R(F'', M) \to \\
\to \mathrm{Tor}_{n+1}^R(F', M) \to \mathrm{Tor}_{n+1}^R(F, M) \to \mathrm{Tor}_{n+1}^R(F'', M) \to \dots
\end{aligned}$$

Now, since F'' is flat, $\mathrm{Tor}_n^R(F'', M) = 0$ for $n \geq 1$ (Theorem 3.11 in Lang, page 622), so the above can be rewritten as

$$\dots \to \mathrm{Tor}_n^R(F', M) \to \mathrm{Tor}_n^R(F, M) \to 0 \to \mathrm{Tor}_{n+1}^R(F', M) \to \mathrm{Tor}_{n+1}^R(F, M) \to 0 \to \dots$$

If F is flat, then $\mathrm{Tor}_n^R(F, M) = 0$ for $n \geq 1$, so the sequence becomes

$$\dots \to \mathrm{Tor}_n^R(F', M) \to 0 \to 0 \to \mathrm{Tor}_{n+1}^R(F', M) \to 0 \to 0 \to \dots$$

Since this sequence is exact, this implies that $\mathrm{Tor}_n^R(F', M) = 0$ for $n \geq 1$, so F' is flat (again by Theorem 3.11). By a similar argument, if F' is flat, we get an exact sequence

$$\dots \to 0 \to \mathrm{Tor}_n^R(F, M) \to 0 \to 0 \to \mathrm{Tor}_{n+1}^R(F, M) \to 0 \to \dots$$

which implies that $\mathrm{Tor}_n^R(F, M) = 0$ for $n \geq 1$, so F is flat. Hence F' is flat if and only if F is flat. $\square$

Proposition 0.10 (Exercise 3b, injective part). *Fix a commutative ring R, and let*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

be an exact sequence of R-modules, where F' is injective. Then F is injective if and only if F'' is injective.

Proof. Fix an R-module M and a projective resolution of M

$$\dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Then we have the following commutative diagram, which is a short exact sequence of chain complexes with morphisms of degree zero.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_0, F') & \longrightarrow & \mathrm{Hom}_R(P_1, F') & \longrightarrow & \mathrm{Hom}_R(P_2, F') & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_0, F) & \longrightarrow & \mathrm{Hom}_R(P_1, F) & \longrightarrow & \mathrm{Hom}_R(P_2, F) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_0, F'') & \longrightarrow & \mathrm{Hom}_R(P_1, F'') & \longrightarrow & \mathrm{Hom}_R(P_2, F'') & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Each column is a short exact sequence, because P_i is projective, so the functor $\mathrm{Hom}_R(P_i, -)$ is exact. Thus the above is a short exact sequence of complexes with morphisms of degree zero. For each row, the nth homology is $\mathrm{Ext}_n^R(M, F'), \mathrm{Ext}_n^R(M, F)$, or $\mathrm{Ext}_n^R(M, F'')$ respectively. By Theorem 2.1 (Lang, page 768), we get a long exact sequence on homology,

$$\begin{aligned}\dots \to \mathrm{Ext}_n^R(M, F') \to \mathrm{Ext}_n^R(M, F) \to \mathrm{Ext}_n^R(M, F'') \to \\ \to \mathrm{Ext}_{n+1}^R(M, F') \to \mathrm{Ext}_{n+1}^R(M, F) \to \mathrm{Ext}_{n+1}^R(M, F'') \to \dots\end{aligned}$$

Now, since F' is injective, $\mathrm{Ext}_n^R(M, F') = 0$ for $n \geq 1$, so the above can be rewritten as

$$\dots \to 0 \to \mathrm{Ext}_n^R(M, F) \to \mathrm{Ext}_n^R(M, F'') \to 0 \to \mathrm{Ext}_{n+1}^R(M, F) \to \mathrm{Ext}_{n+1}^R(M, F'') \to \dots$$

If F is injective, then $\mathrm{Ext}_n^R(M, F) = 0$ for $n \geq 1$, so the sequence becomes

$$\dots \to 0 \to 0 \to \mathrm{Ext}_n^R(M, F'') \to 0 \to 0 \to \mathrm{Ext}_{n+1}^R(M, F'') \to \dots$$

Since this sequence is exact, this implies that $\mathrm{Ext}_n^R(M, F'') = 0$ for $n \geq 1$, so F'' is injective. By a similar argument, if F'' is injective, we get an exact sequence

$$\dots \to 0 \to \mathrm{Ext}_n^R(M, F) \to 0 \to 0 \to \mathrm{Ext}_{n+1}^R(M, F) \to 0 \to \dots$$

which implies that $\mathrm{Ext}_n^R(M, F) = 0$ for $n \geq 1$, so F is injective. Hence F'' is injective if and only if F is injective. □

Proposition 0.11 (Exercise 3b, projective part)**.** *Fix a commutative ring R, and let*

$$0 \longrightarrow F' \xrightarrow{\ \phi\ } F \xrightarrow{\ \psi\ } F'' \longrightarrow 0$$

be an exact sequence of R-modules, where F'' is projective. Then F is projective if and only if F' is projective.

Proof. Fix an R-module M and an injective resolution of M

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots$$

Then we have the following commutative diagram, which is a short exact sequence of chain complexes with morphisms of degree zero.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(F', I_0) & \longrightarrow & \mathrm{Hom}_R(F', I_1) & \longrightarrow & \mathrm{Hom}_R(F', I_2) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(F, I_0) & \longrightarrow & \mathrm{Hom}_R(F, I_1) & \longrightarrow & \mathrm{Hom}_R(F, I_2) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(F'', I_0) & \longrightarrow & \mathrm{Hom}_R(F'', I_1) & \longrightarrow & \mathrm{Hom}_R(F'', I_2) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Each column is a short exact sequence, because I_i is injective, so the functor $\mathrm{Hom}_R(-, I_i)$ is exact. Thus the above is a short exact sequence of complexes with morphisms of degree zero. For each row, the nth homology is $\mathrm{Ext}_n^R(F', M)$, $\mathrm{Ext}_n^R(F, M)$, or $\mathrm{Ext}_n^R(F'', M)$ respectively. By Theorem 2.1 (Lang, page 768), we get a long exact sequence on homology,

$$\begin{aligned}
\ldots \to \mathrm{Ext}_n^R(F'', M) \to \mathrm{Ext}_n^R(F, M) \to \mathrm{Ext}_n^R(F', M) \to \\
\to \mathrm{Ext}_{n+1}^R(F'', M) \to \mathrm{Ext}_{n+1}^R(F, M) \to \mathrm{Ext}_{n+1}^R(F', M) \to \ldots
\end{aligned}$$

Now, since F'' is projective, $\mathrm{Ext}_n^R(F'', M) = 0$ for $n \geq 1$, so the above can be rewritten as

$$\ldots \to 0 \to \mathrm{Ext}_n^R(F, M) \to \mathrm{Ext}_n^R(F', M) \to 0 \to \mathrm{Ext}_{n+1}^R(F, M) \to \mathrm{Ext}_{n+1}^R(F', M) \to \ldots$$

If F is projective, then $\mathrm{Ext}_n^R(F, M) = 0$ for $n \geq 1$, so the sequence becomes

$$\ldots \to 0 \to 0 \to \mathrm{Ext}_n^R(F', M) \to 0 \to 0 \to \mathrm{Ext}_{n+1}^R(F', M) \to \ldots$$

Since this sequence is exact, this implies that $\mathrm{Ext}_n^R(F', M) = 0$ for $n \geq 1$, so F' is projective. By a similar argument, if F' is projective, we get an exact sequence

$$\ldots \to 0 \to \mathrm{Ext}_n^R(F, M) \to 0 \to 0 \to \mathrm{Ext}_{n+1}^R(F, M) \to 0 \to \ldots$$

which implies that $\mathrm{Ext}_n^R(F, M) = 0$ for $n \geq 1$, so F is projective. Hence F' is projective if and only if F is projective. □

Definition 0.12. *Let R be a ring, and M, N be R-modules. An **extension** of M by N is an exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

We define a map from extensions of M by N to $\operatorname{Ext}^1_R(M,N)$. Choose an extension $0 \to N \to E \to M \to 0$. Let P be a projective R-module, with a surjective homomorphism $p : P \to M$. Let $K = \ker p$. Then there is an exact sequence

$$0 \longrightarrow K \xrightarrow{w} P \xrightarrow{p} M \longrightarrow 0$$

where w is the inclusion. Because P is projective, there exists a homomorphism $u : P \to E$, and depending on u a unique homomorphism $v : K \to N$ so that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{w} & P & \xrightarrow{p} & M & \longrightarrow & 0 \\
 & & {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

On the other hand, we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(P,N) \longrightarrow \operatorname{Hom}_R(K,N) \longrightarrow \operatorname{Ext}^1_R(M,N) \longrightarrow 0$$

with the last term on the right being equal to zero because $\operatorname{Ext}^1_R(P,N) = 0$. To the extension $0 \to N \to E \to M \to 0$ we associate the image of v in $\operatorname{Ext}^1_R(M,N)$.

Proposition 0.13 (Exercise 4, Lang Ch. XX Exercise 27)**.** *The association above is a bijection between isomorphism classes of extensions and* $\operatorname{Ext}^1_R(M,N)$.

Proof. Denote the association defined above by Φ. We will construct an inverse for Φ. Let $e \in \operatorname{Ext}^1_R(M,N)$. We have an exact sequence

$$0 \longrightarrow K \xrightarrow{w} P \xrightarrow{p} M \longrightarrow 0$$

where P is projective, $K = \ker p$, and w is the inclusion. Then the following sequence is exact,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}_R(M,N) & \longrightarrow & \operatorname{Hom}_R(P,N) & \longrightarrow & \operatorname{Hom}_R(K,N) \\
 & \longrightarrow & \operatorname{Ext}^1_R(M,N) & \longrightarrow & \operatorname{Ext}^1_R(P,N) & \longrightarrow & \cdots
\end{array}$$

But $\operatorname{Ext}^1_R(P,N) = 0$ since P is projective, so $\operatorname{Hom}_R(K,N) \to \operatorname{Ext}^1_R(M,N)$ is surjective. That is, there exists $v \in \operatorname{Hom}_R(K,N)$ so that $\Phi(v) = e$.

Now define

$$J = \{(v(x), -w(x)) \in N \oplus P : x \in K\}$$

and then define $E = (N \oplus P)/J$. We claim that the map $\psi : N \to E$ given by $y \mapsto (y,0)$ mod J is injective.

$$\ker \psi = \{y \in N : (y,0) \in J\} = \{y \in N : \exists x \in K \; y = v(x), 0 = w(-x)\}$$

Since w is injective, $w(-x) = 0$ implies $x = 0$, so the kernel of ψ is trivial. Hence ψ is injective as claimed. Now we claim that the map $N \oplus P \to M$ given by $(x,y) \mapsto (0, p(y))$ vanishes on J. This follows because $p \circ w = 0$.

$$(v(x), -w(x)) \mapsto (0, p(-w(x)) = (0, -p(w(x))) = (0,0)$$

Hence we have an induced map $E = (N \oplus P)/J \to M$, which is surjective. Thus, from $e \in \mathrm{Ext}^1_R(M,N)$, we have produced an extension $0 \to N \to E \to M \to 0$. We'll denote this association Ψ.

We claim that $\Phi \circ \Psi$ is the identity on $\mathrm{Ext}^1_R(M,N)$. Let $e \in \mathrm{Ext}^1_R(M,N)$. Then $\Psi(e)$ is the extension $0 \to N \to E \to M \to 0$, which, by construction, lives inside the following diagram:

$$\begin{CD}
0 @>>> K @>w>> P @>p>> M @>>> 0\\
@. @VvVV @VVV @V\mathrm{id}VV @.\\
0 @>>> N @>>> (N\oplus P)/J @>>> M @>>> 0
\end{CD}$$

By definition of Φ, $\Phi(\Psi(e))$ is the image of v in $\mathrm{Ext}^1_R(K,N)$, which is e by construction of Ψ. Thus $\Phi \circ \Psi$ is the identity on $\mathrm{Ext}^1_R(M,N)$.

Now we claim that $\Psi \circ \Phi$ maps an extension to an isomorphic extension. Let $0 \to N \to E \to M \to 0$ be an extension of M by N. By definition, its image under Φ is found by choosing a projective module P containing M and constructing the following diagram, and taking the image of v in $\mathrm{Ext}^1_R(M,N)$.

$$\begin{CD}
0 @>>> K @>w>> P @>p>> M @>>> 0\\
@. @VvVV @VuVV @V\mathrm{id}VV @.\\
0 @>>> N @>>> E @>>> M @>>> 0
\end{CD}$$

Denote this image by e. Then by construction of Ψ, $\Psi(e)$ is the extension $0 \to N \to (N \oplus P)/J \to M \to 0$ so that the following diagram commutes.

$$\begin{CD}
0 @>>> K @>w>> P @>p>> M @>>> 0\\
@. @VvVV @VVV @V\mathrm{id}VV @.\\
0 @>>> N @>>> (N\oplus P)/J @>>> M @>>> 0
\end{CD}$$

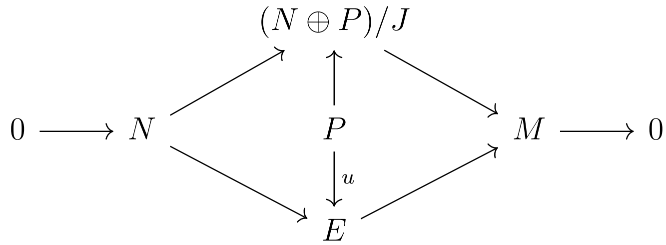
Combining these diagrams,

$$\begin{CD}
0 @>>> N @>>> (N\oplus P)/J @>>> M @>>> 0\\
@. @AAA @AAA @AAA @.\\
0 @>>> K @>w>> P @>p>> M @>>> 0\\
@. @VvVV @VuVV @V\mathrm{id}VV @.\\
0 @>>> N @>>> E @>>> M @>>> 0
\end{CD}$$

then deleting some now irrelevant pieces,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & (N\oplus P)/J & \longrightarrow & M & \longrightarrow & 0\\
 & & \Big\downarrow \mathrm{id} & & \Big\uparrow & & \Big\downarrow \mathrm{id} & & \\
 & & & & P & & & & \\
 & & & & \Big\downarrow u & & & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and further collapsing, we get



Then we get a map $(N \oplus P)/J \to E$ by $(x, y) \mapsto u(y)$. This map makes the above diagram commute, so these extensions are isomorphic. $\square$

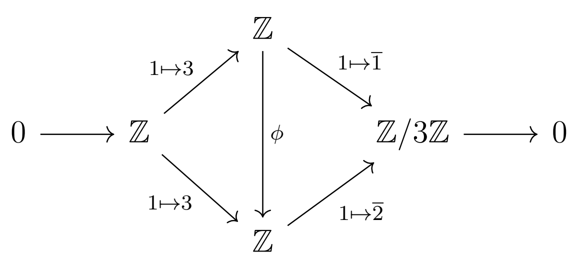
Proposition 0.14 (Exercise 5)**.** *Three inequivalent extensions of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}$ are given by*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1 \mapsto 3} & \mathbb{Z} & \xrightarrow{1 \mapsto \overline{1}} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1 \mapsto 3} & \mathbb{Z} & \xrightarrow{1 \mapsto \overline{2}} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1 \mapsto (1,0)} & \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z}) & \xrightarrow[(0,1) \mapsto \overline{1}]{(1,0) \mapsto \overline{0}} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Proof. First we check that these are all exact sequences.

$$\begin{aligned} \mathrm{im}(1 \mapsto 3) &= 3\mathbb{Z} = \ker(1 \mapsto \overline{1}) \\ \mathrm{im}(1 \mapsto 3) &= 3\mathbb{Z} = \ker(1 \mapsto \overline{2}) \\ \mathrm{im}(1 \mapsto (1,0)) &= \{(n, 0) : n \in \mathbb{Z}\} = \ker((1,0) \mapsto 0) \end{aligned}$$

The third extension is not equivalent to the first two because there is no isomorphism $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, since the former is torsion free and the latter is not. If the first two are equivalent extensions, then there is an isomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ so that the following diagram commutes:



Just looking at the left triangle implies that $\phi = \mathrm{Id}_{\mathbb{Z}}$, which measn that the right triangle does not commute. Hence no such isomorphism exists, so these are inequivalent extensions. $\square$

Proposition 0.15 (Exercise 6a)**.** *Let $R = \mathbb{C}[x, y]$ be the ring of polynomials with complex coefficients in two variables x, y. Define $R \times \mathbb{C} \to \mathbb{C}$ by*

$$f \cdot z \mapsto (f(0,0))(z)$$

then $\mathbb{C}$ is an R-module with this structure.

Proof. Let $z, z_1, z_2 \in \mathbb{C}$ and $f, g \in \mathbb{C}[x, y]$. Let p_1 denote the constant polynomial $p_1(x, y) = 1$, which is the multiplicative unit in $\mathbb{C}[x, y]$.

$$\begin{aligned} f \cdot (z_1 + z_2) &= f(0,0)(z_1 + z_2) = f(0,0)z_1 + f(0,0)z_2 = f \cdot z_1 + f \cdot z_2 \\ (f+g) \cdot z &= (f+g)(0,0)z = (f(0,0) + g(0,0))z = f(0,0)z + g(0,0)z = f \cdot z + g \cdot z \\ (fg) \cdot z &= (fg)(0,0)z = (f(0,0)g(0,0))z = f(0,0)(g \cdot z) = f \cdot (g \cdot z) \\ p_1 \cdot z &= p_1(0,0)z = z \end{aligned}$$

Thus this gives $\mathbb{C}$ an R-module structure. □

Proposition 0.16 (Exercise 7a)**.** *Let R be a principal ideal domain, and let F be an R-module. Then F is flat if and only if it is torsion free.*

Proof. Suppose F is flat. For $r \in R$, define $\psi_r : R \to R$ by $\psi_r(x) = rx$. Since R is an integral domain, ψ_r is injective. Since F is flat, the induced map $\mathrm{Id}_F \otimes \psi_r : F \otimes R \to F \otimes R$ given by $f \otimes x \mapsto f \otimes \psi_r(x)$ is injective. Suppose f is a torsion element of F, that is, there exists $r \in R$ with $r \neq 0$ so that $rf = 0$. Then

$$\begin{aligned} \mathrm{Id}_F \otimes \psi_r(f \otimes 1) &= f \otimes r = (rf) \otimes 1 = 0 \\ \mathrm{Id}_F \otimes \psi_r(0 \otimes 1) &= 0 \otimes r = 0 \end{aligned}$$

By injectivity of $\mathrm{Id}_F \otimes \psi_r$, this implies $f \otimes 1 = 0 \otimes 1$. Recall that the map $F \otimes R \to F$ given by $f \otimes 1 \to f$ is an isomorphism, so $f = 0$. Thus any torsion element of f is zero, so F is torsion free.

Now suppose F is torsion free. Recall that F is flat if and only if $\mathrm{Tor}_1^R(F, R/I) = 0$ for every ideal $I \subset R$. Let $I \subset R$ be an ideal. Since R is a PID, $I = (x)$ for some $a \in R$. By Proposition 0.1,

$$\mathrm{Tor}_1^R(F, R/(x)) = \{f \in F : xf = 0\}$$

which is zero because F is torsion free. Thus F is flat. □

Lemma 0.17 (for Exercise 7b)**.** *A $\mathbb{Z}$-module is projective if and only if it is free.*

Proof. Free modules are always projective, over any ring, so one direction is done. Let M be a projective $\mathbb{Z}$-module. Then M is a direct summand of a free $\mathbb{Z}$-module, that is, there exists an abelian group N so that $M \oplus N$ is free abelian. Then M is a subgroup of a free abelian group, so it is free abelian (Theorem 7.3 in Lang). Thus M is a free $\mathbb{Z}$-module. □

Lemma 0.18 (for Exercise 7b)**.** *$\mathbb{Q}$ is not a free $\mathbb{Z}$-module.*

Proof. Suppose that $\mathbb{Q}$ is free as a $\mathbb{Z}$-module. Then the basis cannot have just one element x, since then $\frac{x}{2}$ would not be an integer multiple of x, and then x would not generate $\mathbb{Q}$ over $\mathbb{Z}$. Thus there must be a linearly independent set with at least two nonzero elements, say $\frac{a}{b}$ and $\frac{c}{d}$ with $a, b, c, d \in \mathbb{Z} \setminus \{0\}$. Then note that $\frac{\mathrm{lcm}(a,c)}{a}$ and $\frac{\mathrm{lcm}(a,c)}{c}$ are both integers, so

$$\left(\frac{b\,\mathrm{lcm}(a,c)}{a}\right)\left(\frac{a}{b}\right) + \left(\frac{-d\,\mathrm{lcm}(a,c)}{c}\right)\left(\frac{c}{d}\right) = 0$$

is a nontrivial linear combination. Thus there are no linearly independent subset of $\mathbb{Q}$ with at least two elements. Thus there can be no basis for $\mathbb{Q}$ over $\mathbb{Z}$, so it is not free. □

Proposition 0.19 (Exercise 7b)**.** *As a $\mathbb{Z}$-module, $\mathbb{Q}$ is flat but not projective.*

Proof. Since $\mathbb{Q}$ is torsion-free and $\mathbb{Z}$ is a PID, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module by part (a). By the previous two lemmas, a $\mathbb{Z}$ module is free if and only if it is projective, and $\mathbb{Q}$ is not free; hence $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. □